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Dirac Equation in Space–Time With Torsion

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The Dirac equation in a curved space–time endowed with compatible affine connection is reconsidered. After a detailed decomposition of the total action, the equation is obtained by varying with respect to the Dirac spinor and the torsion field. The result is a known Dirac-like equation with constraints that can be interpreted as the equation of a self-interacting spin 1/2 particle in curved space–time. The scheme is then translated into the language of the 2-spinor formalism of curved space–time based on the choice of a null tetrad frame. The spinorial equation so obtained coincides with the standard one in case of no torsion, while in general it remains a nonlinear equation describing a self-interacting spin 1/2 particle. The nonlinearity is produced by the interaction of the particle with its own current that remains conserved as in the free torsion case.

KEY WORDS: Dirac equation; tensors; spinors; torsion.

1. INTRODUCTION

Among the different formulations of the Dirac equation in curved space–time, the one that includes the torsion is of interest. This gives additional degrees of freedom to the theory that could be used to describe further physical interactions. Torsion effects were already considered by Gürsey (1957) and Finkelstein (1960) (see references in these papers). They both found a nonlinear spinor equation, even starting from slightly different assumptions of uniform torsion. Nonlinear terms induced by torsion in the Dirac equation have been discussed also by Hehl and Datta (1971) and Hehl *et al.* (1976) (see references therein) in their general study of spinor equations in general relativity.

Recently the Dirac equation with and without torsion and under different approximations has been employed to study neutrino oscillations in curved space–time. To mention some results, we recall, for instance, that Piriz *et al.* (1996) have focused on the case of strong gravitational field; Cardall and Fuller (1997) have

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developed a study of the gravitational contributions to neutrino spin precession in presence of magnetic field. Alimohammadi and Shariati (1999) extended the study by Cardall and Fuller to explore the effect of torsion on neutrino oscillations; similarly Zhang (2000) developed the formalism in case of zero curvature but of nonzero torsion space–time. The theoretical scheme adopted for the description of a spin 1/2 particle in space–time with torsion is based on a Lagrangian, invariant both on general coordinates and on local Lorentz rotations, whose explicit expression can be found, for instance, in the books by Nakahara (1990) and Buchbinder *et al.* (1992) (compare also with Weinberg, 1972). The treatment is generally done in the four-dimensional spinor formalism and is based on an orthogonal tetrad.

Since the two spinor version of this theory seems to lack in the literature, it is of interest to have the counterpart of it in the language of Newman and Penrose (1962) formalism. Of course this could be done by promoting the spinor covariant derivative of the spinorial Dirac equation in curved space–time (as it appears, for instance in the book by Penrose and Rindler, 1983) to include torsion. However, as a consequence of general properties of the affine connection, this amounts to add to the standard covariant spinor derivative a spinor term whose physical interpretation seems difficult to define a priori.

In order to translate the scheme into the language of two-dimensional spinor formalism, some basic definitions and properties of space–time with affine connection are first recollected. This leads to the explicit form of the Einstein–Hilbert–Cartan action. The expression of the Dirac action is developed as far as possible by considering all its terms, also those sometimes neglected in physical applications. The field equations are obtained by varying the total action with respect to the Dirac spinor and the torsion field. There comes out a Dirac-like equation with constraints that can be interpreted as a nonlinear Dirac equation in curved space–time describing a self-interacting spin 1/2 particle.

The result is then transformed into the two-dimensional spinor formalism by using standard representation of the Dirac matrices on general null tetrad frames. The result is checked in the torsion free case: one obtains exactly the Dirac equation as written in Chandrasekhar's book (1983) in terms of spin coefficients and directional derivatives. In case of nonzero torsion the nonlinear terms can be interpreted as the interaction of the particle with its own spinorial current. It comes out that the current is conserved as in the torsion free case.

2. SPACE–TIME WITH TORSION

In the following the space–time is assumed to be a four-dimensional Lorentz manifold (M, g) endowed by an affine connection ∇ (we refer to Nakahara (1990) for notations and mathematical conventions). The affine connection ∇, whose affine coefficients are denoted $\Gamma^{\lambda}_{\mu\nu}$, is required to realize a *metric compatible* connection $\nabla_\lambda g_{\mu\nu} = 0$ that is

$$\partial g_{\mu\nu} - \Gamma^\kappa_{\lambda\mu} g_{\kappa\nu} - \Gamma^\kappa_{\lambda\nu} g_{\mu\kappa} = 0. \tag{1}$$

The Levi–Civita connection, denoted $\tilde{\nabla}$ and acting on the same space–time, is the one arising from the metric tensor through the Christoffel coefficients $\{^{\ \kappa}_{\lambda\nu}\} = \frac{1}{2} g^{\kappa\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$. A central point in the mathematical theory is that the relation between ∇ and $\tilde{\nabla}$ is expressed by

$$\Gamma^\kappa_{\mu\nu} = \left\{^{\ \kappa}_{\mu\nu}\right\} + K^\kappa_{\mu\nu} \tag{2}$$

where the contorsion tensor $K^\kappa_{\mu\nu}$ is defined through the torsion tensor $T^\kappa_{\mu\nu} = \Gamma^\kappa_{\mu\nu} - \Gamma^\kappa_{\nu\mu}$ by means of $K^\kappa_{\mu\nu} \equiv \frac{1}{2}(T^\kappa_{\mu\nu} + T^\kappa_{\mu\nu} + T^\kappa_{\nu\mu})$. Since also $\tilde{\nabla}_\mu g_{\alpha\beta} = 0$, the decomposition (2) inserted in (1) implies the symmetry property of the contorsion tensor

$$K_{\lambda\mu\nu} = -K_{\nu\mu\lambda}. \tag{3}$$

As in the free torsion case, the Riemann curvature tensor $R^\kappa_{\lambda\mu\nu}$ can be entirely expressed in terms of the affine connection coefficients

$$R^\kappa_{\lambda\mu\nu} = \partial_\mu \Gamma^\kappa_{\nu\lambda} - \partial_\nu \Gamma^\kappa_{\mu\lambda} + \Gamma^\eta_{\nu\lambda}\Gamma^\kappa_{\mu\eta} - \Gamma^\eta_{\mu\lambda}\Gamma^\kappa_{\nu\eta} \tag{4}$$

As a consequence of the decomposition (2), the Riemann curvature tensor itself can be separated into a part completely expressed by the Christoffel symbols and a part containing the contorsion. The same decomposition property holds then for the Ricci tensor $R^\lambda_{\mu\lambda\nu}$ and for the scalar curvature $R = g^{\mu\nu} R^\lambda_{\mu\lambda\nu}$. To make the results more clear and for the following purposes it is convenient to preliminary decompose (see, e.g., Alimohammadi and Shariati, 1999) also the contorsion in the form

$$K_{\alpha\mu\nu} = \frac{1}{3}(g_{\alpha\mu}\tau_\nu - g_{\nu\mu}\tau_\alpha) + \frac{1}{2}\mathcal{A}^\sigma \epsilon_{\sigma\alpha\mu\nu} + U_{\alpha\mu\nu} \tag{5}$$

where it has been defined

$$\tau_\mu = g^{\alpha\beta} K_{\alpha\beta\mu}, \qquad \mathcal{A}^\sigma = \frac{1}{3}\epsilon^{\sigma\alpha\beta\mu} K_{\alpha\beta\mu}. \tag{6}$$

The term $U_{\alpha\mu\nu}$ is infact determined by Eq. (5) and it has therefore the properties

$$U_{\alpha\mu\nu} = -U_{\nu\mu\alpha}, \qquad g^{\alpha\mu} U_{\alpha\mu\nu} = 0, \qquad \epsilon^{\sigma\alpha\mu\nu} U_{\alpha\mu\nu} = 0. \tag{7}$$

By using Eqs. (2), (5–7) into the expression (4) one gets, with some rearrangements

$$R = \tilde{R} - \frac{2}{\sqrt{g}}\partial_\kappa(\sqrt{g}\tau^\kappa) - \frac{1}{3}\tau^2 + \frac{3}{2}\mathcal{A}^2 + U_{\alpha\mu\nu}U^{\mu\alpha\nu} \tag{8}$$

where $g = |\det g_{\mu\nu}|$ and $\tilde{R}$ is the part expressed in terms of the Chrisoffel symbols alone not containing the contorsion tensor. The field equations can then be obtained by varying the Einstein–Hilbert–Cartan action

$$S_{\text{EHC}} = \int d^4x\sqrt{g}\left(\tilde{R} - \frac{1}{3}\tau^2 + \frac{3}{2}\mathcal{A}^2 + U_{\alpha\mu\nu}U^{\mu\alpha\nu}\right) \tag{9}$$

with respect to the metric and the contortion tensor fields. The divergence term in (8) has been neglected in the action because no variations of the boundary will be considered.

3. DIRAC EQUATION WITH TORSION

The description of a Dirac spinor ψ in the four-dimensional Lorentz manifold M can be done in general by a well-known scalar action whose Lagrangian is a scalar both under coordinated change and Lorentz rotations. To do this let e_a^μ be a local reference frame (tetrad), with associated inverse matrix e_ν^b, defined by (Nakahara, 1990; Weinberg, 1972)

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}; \qquad \eta_{ab} = e_a^\mu e_b^\nu g_{\mu\nu} \tag{10}$$

$\eta_{ab} = \eta^{ab} = \text{diag}\,(1, -1, -1, -1)$ the Minkowski metric. Tetrad indices are denoted by latin letters, coordinate indices by greek letters. Given the Dirac matrices satisfying $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ the γ^μ matrices defined by $\gamma^\mu = e_a^\mu\gamma^a$ satisfy the relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. The mentioned action is then given by (Nakahara, 1990; Weinberg, 1972)

$$S_{\text{D}} = \int d^4x\sqrt{g}\bar{\psi}\left[i\gamma^a e_a^\mu(\partial_\mu + \Omega_\mu) + m\right]\psi. \tag{11}$$

As usual $\bar{\psi} = \psi^+\gamma^o$. The term involving the spin connection Ω_μ is defined by

$$\gamma^\mu\Omega_\mu = -\frac{1}{8}e_a^\mu e_b^\nu(\nabla_\mu e_{c\nu})\gamma^a[\gamma^b, \gamma^c]. \tag{12}$$

By considering that (Cardall and Fuller, 1997)

$$\gamma^a[\gamma^b, \gamma^c] = 2\eta^{ab}\gamma^c - 2\eta^{ac}\gamma^b - 2i\gamma_5\gamma_d\epsilon^{dabc} \tag{13}$$

where $\gamma_5 = \text{diag}\,(I_2, -I_2)$, $\{\gamma^a, \gamma_5\} = 0$ and by writing alternatively

$$\nabla_\mu e_{c\nu} = \tilde{\nabla}_\mu e_{c\nu} - K_{\nu\mu}^{\lambda} e_{c\lambda} \tag{14}$$

$$\tilde{\nabla}_\mu e_{c\nu} = \frac{e_{c\nu,\mu} + e_{c\mu,\nu}}{2} + \frac{e_{c\nu,\mu} - e_{c\mu,\nu}}{2} - \left\{{}^{\lambda}_{\nu\,\mu}\right\}e_{c\lambda} \tag{15}$$

the product in (12) can be developed by taking into account symmetry properties in the exchange of some of the indices involved. One obtains

$$\gamma^a e_a^\mu\Omega_\mu = \frac{i}{4}\gamma^\mu\gamma_5 e_{b\mu}(e_{c\nu,\sigma} - e_{c\sigma,\nu})\epsilon^{bcad}e_a^\mu e_d^\sigma - \frac{1}{2}\gamma^\lambda e_\lambda^c\tilde{\nabla}_\nu e_c^\nu + \frac{1}{4}\gamma^\lambda\tau_\lambda + \frac{3}{4}i\gamma^\lambda\gamma_5\mathcal{A}_\lambda \tag{16}$$

Therefore the Dirac action can be written more explicitly

$$S_{\rm D} = \int d^4x\sqrt{g}\bar{\psi}\{i\gamma^\mu[\partial_\mu + i\gamma_5(A^G_{1\mu} + A^T_{1\mu}) + A^G_{2\mu} + A^T_{2\mu}] + m\}\psi \qquad (17)$$

where

$$\begin{aligned} A^G_{1\mu} &= \frac{1}{4}e_{b\mu}(e_{c\nu,\sigma} - e_{c\sigma,\nu})\epsilon^{bcad}e^\mu_a e^\sigma_d, \qquad & A^T_{1\mu} &= \frac{3}{4}\mathcal{A}_\mu, \\ A^G_{2\mu} &= -\frac{1}{2}e^c_\mu\tilde{\nabla}_\alpha e^\alpha_c, & A^T_{2\mu} &= \frac{1}{4}\tau_\mu. \end{aligned} \qquad (18)$$

Alternatively $A^G_{1\mu}$ and $A^G_{2\mu}$ can be expressed in terms of the spin coefficients (Ricci rotation coefficients, see Chandrasekhar, 1983) defined by $\gamma_{abc} = e^\nu_a(\tilde{\nabla}_\mu e_{b\nu})e^\mu_c$. From Eqs. (12)–(14) one readily finds

$$A^G_{1\mu} = -\frac{1}{4}\epsilon^{dabc}\gamma_{bca}e_{d\mu}, \qquad A^G_{2\mu} = -\frac{1}{2}\gamma^{ad}{}_a e_{d\mu}. \qquad (19)$$

This form will be useful when passing to two-dimensional spinors. The term containing torsion has been distinguished by the label T the remaining terms by G.

The field equation of the spin 1/2 particle coupled to gravity can now be obtained by varying the total action $S = S_{\rm D} + S_{\rm EHC}$ where $S_{\rm D}$, $S_{\rm EHC}$ are given by (17) and (9), with respect to the fields $\bar{\psi}$, $U_{\alpha\mu\nu}$, τ_μ, $\mathcal{A}^\mu$. One obtains

$$\gamma^\mu[\partial_\mu + i\gamma_5(A^G_{1\mu} + A^T_{1\mu}) + A^G_{2\mu} + A^T_{2\mu}]\psi = im\psi \qquad (20)$$

$$U_{\alpha\mu\nu} = 0 \qquad (21)$$

$$\tau^\mu = \frac{3}{8}i\bar{\psi}\gamma^\mu\psi \qquad (22)$$

$$\mathcal{A}^\mu = \frac{1}{4}\bar{\psi}\gamma_5\gamma^\mu\psi \qquad (23)$$

(Note that the absense of particle gives $\tau_\mu = \mathcal{A}_\mu = 0$ so that no nontrivial torsion is possible in this case.) The Eq. (20) is subjected to the constraint (22) and (23). By taking into account the definitions (18), the equation can be written

$$\gamma^\mu\left[\partial_\mu + i\gamma_5 A^G_{1\mu} + A^G_{2\mu} + \frac{3}{16}i\bar{\psi}\gamma_\mu\gamma_5\psi + \frac{3}{32}i\bar{\psi}\gamma_\mu\psi\right]\psi = im\psi. \qquad (24)$$

It is possible to translate the previous results into the languages of the two-dimensional spinors formalism of Newmann and Penrose, 1962.

4. DIRAC EQUATION IN TWO SPINOR FORM

The tetrad e_a^μ of Eq. (10) is now chosen to be

$$\eta_{ab} \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{25}$$

One can then check that by mimicking the procedure of the previous section, the same final results (20), (17) still hold, the only difference being that e_a^μ is now a null tetrad frame. To simplify notations it is convenient to denote $e_1^\mu \equiv l^\mu$, $e_2^\mu \equiv n^\mu$, $e_3^\mu \equiv m^\mu$, $e_4^\mu \equiv m^{\star\mu}$ (with l^μ, n^μ real; $m^{\star\mu} = m^{\mu\star}$). Therefore $l^\mu l_\mu = n^\mu n_\mu = m^\mu m_\mu = m^{\star\mu} m^\star_\mu = 0, l^\mu n_\mu = 1, m^\mu m^\star_\mu = -1$. Associated to the null tetrad frame there are then the 4×4 Dirac matrices

$$\gamma_\mu = \sqrt{2}\begin{pmatrix} 0 & G_\mu \\ -G^\star_\mu & 0 \end{pmatrix}, \quad \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} I_4 \tag{26}$$

($G^\star$ is the adjoint of G, see e.g. Penrose and Rindler, 1984) that still satisfy the usual anticommutation relations. This is a consequence of the definition of the G matrices in terms of the spin matrices

$$G_\mu \equiv \sigma^{A}_{\mu B'} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} m^\star_\mu & n_\mu \\ -l_\mu & -m_\mu \end{pmatrix} \tag{27}$$

that in their turn satisfy the relations $G_\mu G^\star_\nu + G_\nu G^\star_\mu = -2g_{\mu\nu} I_2$. By further setting $\psi \equiv \binom{Q^A}{\bar{P}^{B'}}$ and from $\gamma_5 = -i I_4$ one has

$$\begin{aligned} \sigma^\mu_{BA'}\partial_\mu P^B + \sigma^\mu_{BA'}\left(A^{G\star}_{1\mu} + A^{T\star}_{1\mu} - A^{G\star}_{2\mu} - A^{T\star}_{2\mu}\right)P^B &= -i\frac{m}{\sqrt{2}}\bar{Q}_{A'} \\ \sigma^\mu_{BA'}\partial_\mu Q^B + \sigma^\mu_{BA'}\left(A^{G}_{1\mu} + A^{T}_{1\mu} - A^{G}_{2\mu} - A^{T}_{2\mu}\right)Q^B &= -i\frac{m}{\sqrt{2}}\bar{P}_{A'}. \end{aligned} \tag{28}$$

The expressions $\sigma^\mu_{BA'}\partial_\mu = \partial_{BA'}$ are the directional derivatives generally denoted $\partial_{00'} = D = l^\mu \partial_\mu$, $\partial_{01'} = \delta = m^\mu \partial_\mu$, $\partial_{10'} = \delta^\star = m^{\star\mu}\partial_\mu$, $\partial_{11'} = \Delta = n^\mu \partial_\mu$.

We first consider the torsion free case. Denoting as usual the spin coefficients (Chandrasekhar, 1983)

$$\begin{aligned} &\rho = \gamma_{314} \quad \epsilon = \frac{1}{2}(\gamma_{211} + \gamma_{341}) \quad \pi = \gamma_{241} \quad \alpha = \frac{1}{2}(\gamma_{214} + \gamma_{344}) \\ &\mu = \gamma_{243} \quad \gamma = \frac{1}{2}(\gamma_{212} + \gamma_{342}) \quad \tau = \gamma_{312} \quad \beta = \frac{1}{2}(\gamma_{213} + \gamma_{343}) \end{aligned} \tag{29}$$

one can simplify the expressions

$$\sqrt{2}\sigma^{\mu}_{BA'}\left(A^G_{1\mu} - A^G_{2\mu}\right) = \begin{pmatrix} -\delta_{d2} & \delta_{d4} \\ \delta_{d3} & -\delta_{d1} \end{pmatrix} \left(\frac{1}{4}\epsilon^{dabc}\gamma_{bca} - \frac{1}{2}\gamma^{ad}{}_{a}\right)$$
$$= \begin{pmatrix} \epsilon - \rho & \beta - \tau \\ \pi - \alpha & \mu - \gamma \end{pmatrix} \tag{30}$$

(δ_{di} is the Kronecker delta). The Dirac Eq. (28) then becomes in terms of the spin coefficients and directional derivatives

$$\begin{aligned} (\sqrt{2}D + \epsilon^\star - \rho^\star)P^o + (\sqrt{2}\delta^\star + \pi^\star - \alpha^\star)P^1 &= -im\bar{Q}_0 \\ (\sqrt{2}\delta + \beta^\star - \tau^\star)P^o + (\sqrt{2}\Delta + \mu^\star - \gamma^\star)P^1 &= -im\bar{Q}_1 \\ (\sqrt{2}D + \epsilon - \rho)Q^o + (\sqrt{2}\delta^\star + \pi - \alpha)Q^1 &= -im\bar{P}_0 \\ (\sqrt{2}\delta + \beta - \tau)Q^o + (\sqrt{2}\Delta + \mu - \gamma)Q^1 &= -im\bar{P}_1 \end{aligned} \tag{31}$$

that coincides, after redefining $\Omega_\mu \rightarrow \sqrt{2}\Omega_\mu$, with the Dirac equation as considered in Chandrasekhar (1983) (see also Zecca, 1996). Hence it can be compactly written ($\mu_\star = m/\sqrt{2}$)

$$\begin{aligned} \tilde{\nabla}_{AB'}P^A &= -i\mu_\star\bar{Q}_{B'} \\ \tilde{\nabla}_{AB'}Q^A &= -i\mu_\star\bar{P}_{B'} \end{aligned} \tag{32}$$

where $\tilde{\nabla}_{AB'}$ is the conventional covariant spinor derivative in curved space–time (Penrose and Rindler, 1984).

If the torsion does not vanishes, the Eq. (28) can be interpreted as the Dirac equation in the two-dimensional spinor formalism. Equation (28) is a nonlinear equation on account of the terms quadratic in ψ arising from $A^T_{1\mu}$, $A^T_{2\mu}$ through Eqs. (18), (22), and (23). These terms can be further explicited to obtain

$$\begin{aligned} \sigma^{\mu}_{BA'}\left(A^T_{1\mu} - A^T_{2\mu}\right) &= -\frac{9}{32}i\bar{\psi}\gamma^\mu\psi \\ &= -\frac{9}{32}i(P_B\bar{P}_{A'} + Q_B\bar{Q}_{A'}). \end{aligned} \tag{33}$$

Here it has been used $\bar{\psi} \equiv (-P_B, \bar{Q}_{A'})$ (compare also with Illge, 1993). In terms of the spinorial current $J_{BA'} = P_B\bar{P}_{A'} + Q_B\bar{Q}_{A'}$ the Eq. (28) can now be written

$$\begin{aligned} \tilde{\nabla}_{BA'}P^B + icJ_{BA'}P^B &= -i\mu_\star\bar{Q}_{A'} \\ \tilde{\nabla}_{BA'}Q^B - icJ_{BA'}Q^B &= -i\mu_\star\bar{P}_{A'} \quad (c = 9\sqrt{2}/32). \end{aligned} \tag{34}$$

As a consequence of Eq. (34) one can check that the current $J^{BA'}$ is still conserved, $\tilde{\nabla}_{AB'}J^{AB'} = 0$, as in the torsion free case (e.g. Zecca, 1995). It is worth noticing that the two equations in (34) remain coupled (contrarily to the torsion free case) also in the massless case. For what concerns the solution of Eq. (34) there are some particular situations that seem to reduce the difficulty. For instance, in case of static metric it is easily seen that the time dependence of the solution factors out in a form like $\exp(ikt)$. Nevertheless Eq. (34) remains a nonlinear equation difficult to be solved.

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